

Metastability of $(d + n)$ -dimensional elastic manifolds

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We investigate the depinning of a massive elastic manifold with d internal dimensions, embedded in a $(d + n)$ -dimensional space, and subject to an isotropic pinning potential $V(\mathbf{u}) = V(|\mathbf{u}|)$. The tunneling process is driven by a small external force \mathbf{F} . We find the zero temperature and high temperature instantons and show that for the case $1 \leq d \leq 6$ the problem exhibits a sharp transition from quantum to classical behavior: At low temperatures $T < T_c$ the Euclidean action is constant up to exponentially small corrections, while for $T > T_c$, $S_{\text{Eucl}}(d, T)/\hbar = U(d)/T$. The results are universal and do not depend on the detailed shape of the trapping potential $V(\mathbf{u})$. Possible applications of the problem to the depinning of vortices in high- T_c superconductors and nucleation in d -dimensional phase transitions are discussed. In addition, we determine the high-temperature asymptotics of the preexponential factor for the $(1 + 1)$ -dimensional problem.

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I. INTRODUCTION

The decay of metastable states is a basic phenomenon of great generality¹ with numerous applications in a large number of contexts, ranging from the decay of false vacua in field theory², e.g. in cosmology³, to the creep type motion of topological defects in solids^{4,5}. At a given temperature T the inverse lifetime Γ of a metastable system can be written in the form $\Gamma = Ae^{-S_{\text{Eucl}}(T)/\hbar}$, with $S_{\text{Eucl}}(T)$ the Euclidean action of the saddle-point configuration and A the prefactor determined by the associated fluctuations. In this paper we present a calculation of the full temperature dependence of the Euclidean action of a metastable d -dimensional elastic manifold pinned by an isotropic potential $V(\mathbf{u}) = V(|\mathbf{u}|)$ in the presence of a small external force. Typical examples of physical realizations of this model system are a string pinned by a linear or columnar defect ($(1+1)$ -, and $(1+2)$ -dimensional problems respectively) or a membrane pinned by a plane ($(2+1)$ -dimensional problem).

With increasing temperature, the decay changes its nature from quantum to classical. The function $S_{\text{Eucl}}(T)$ might be either a smooth function of temperature or exhibit a sharp kink with a discontinuity in its first derivative at the crossover temperature T_c . The former case is called a “second-order” transition from quantum to classical behavior⁶, while in the latter case a first-order transition takes place. The wording “transition” is appropriate as the crossovers have all the features of mean-field phase transitions upon identifying the Euclidean action with the free energy.

The possibility of a first-order transition in a tunneling problem has first been discussed by Lifshitz and Kagan⁷ in the context of quantum phase transitions in ^4He systems. The general theory and an appropriate criterion for a first-order transition in 1D Hamiltonian systems has been developed by Meshkov⁸ and by Ioselevich and Rashba⁹, see also the paper of Chudnovsky¹⁰. For the case of non-Hamiltonian systems this problem has been considered in Ref. 11. Recently, the possibility of observing a first-order transition from quantum to classical behavior in spin systems has been discussed in Ref. 12.

Systems with many degrees of freedom have been studied by Garriga¹³ (see also Ref.¹⁴) and he finds that, for the case of tunneling of massive $d = 2$ and $d = 3$ elastic manifolds between two minima disbalanced by the action of a small external force, the Euclidean action exhibits a discontinuous derivative, while for a $d = 1$ dimensional string the problem exhibits a smooth crossover¹⁵. A natural counterpart of the bistability problem studied by Coleman, Garriga, and others is the generic problem of weak metastability, see Fig. 1. In this paper we then study the problem where an elastic manifold is pinned by an isotropic potential well and is driven by a weak external force. After the tunneling process (or thermal activation at high temperatures) the manifold becomes free, while in the previous case it is repinned again in the second valley. The approximation introduced by Coleman² and used by Garriga is usually termed the “thin wall approximation”, while in the present problem it is more appropriate to denote the technique used as a “thick wall approximation”.

We find that at least for the case $1 \leq d \leq 6$ the problem exhibits a sharp transition from quantum to classical behavior: The function $S_{\text{Eucl}}(d, T)$ is a constant (up to exponentially small corrections) at temperatures lower than

the critical temperature T_c , while for $T > T_c$, $S_{\text{Eucl}}(d, T)/\hbar = U(d)/T$, with U the activation energy. The (1+1)-dimensional problem has been considered by Skvortsov¹⁶. In the present paper we generalize the results of this work for an arbitrary number of internal degrees of freedom d and improve on the calculation of the preexponential factor for the (1 + 1)-problem as the approximations made in Ref. 16 turn out to be too rough to reach the correct result.

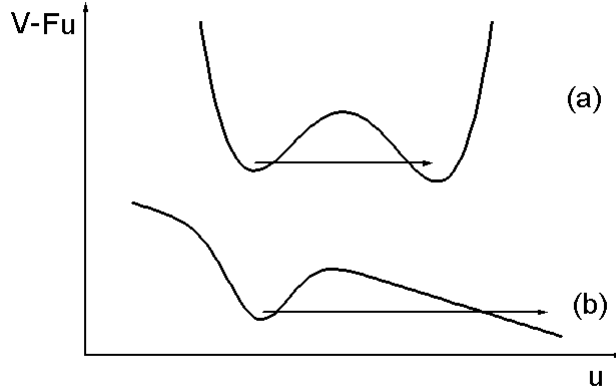


Fig.1 The two generic decay problems of bistability (a) and weak metastability (b).

The paper is organized as follows: in Sec. II A we formulate the problem. In Sec. II B and II C the zero temperature and thermal instantons are found and the corresponding Euclidean actions are calculated. In Sec. III the instability temperature of the thermal instanton is calculated and the full temperature dependence of the Euclidean action is found. Sec. IV is devoted to the calculation of the high-temperature asymptotics of the preexponential factor for the (1 + 1)-problem. We summarize the results in Sec. V and discuss their possible applications to the thermal depinning of vortices in high- T_c superconductors and nucleation phenomena in phase transitions.

II. EUCLIDEAN ACTION AT LOW AND HIGH TEMPERATURES

A. General expression

Consider a massive manifold with d internal dimensions embedded in a $(d + n)$ -dimensional space. The (real time) Lagrangian of the manifold can be written in the form

$$\mathcal{L}[\mathbf{u}(\mathbf{x}, t)] = \int_{\Omega} d\mathbf{x} \left\{ \frac{\rho}{2} \left(\frac{\partial \mathbf{u}}{\partial t} \right)^2 - \frac{\epsilon}{2} \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^2 - V(|\mathbf{u}|) + \mathbf{F} \cdot \mathbf{u} \right\}. \quad (1)$$

Here, $\mathbf{u} \in \mathcal{R}^n$ is the (transverse) displacement field of the manifold, $\mathbf{r} \in \mathcal{R}^d$ is the vector characterizing the internal degrees of freedom, ρ and ϵ are the mass density and the elasticity of the manifold, respectively, and $V(|\mathbf{u}|)$ is an $O(n)$ -invariant trapping potential. The function $V(u)$ is supposed to be monotonously increasing, $V(0) = 0, V(\infty) = V_0$. The integration in Eq. (1) goes over the volume Ω of the manifold. Finally, $\mathbf{F} = (F, 0, \dots, 0)$ is the external driving force which we assume to be small, $F \ll V_0/a$, with a the characteristic radius of the pinning potential $V(u)$. Specific realizations of this model system are strings moving in a plane ((1 + 1)-problem) or in space ((1 + 2)-problem), or membranes pinned by planar boundaries or interfaces in 3D space ((2 + 1)-problems).

After the transformation $t \rightarrow -i\tau$ and $S \rightarrow -iS_{\text{Eucl}}$ the Euclidean action of the manifold reads

$$S_{\text{Eucl}}[\mathbf{u}(\mathbf{x}, \tau)] = \int_{-\hbar/2T}^{+\hbar/2T} d\tau \int_{\Omega} d\mathbf{x} \left\{ \frac{\rho}{2} \left(\frac{\partial \mathbf{u}}{\partial \tau} \right)^2 + \frac{\epsilon}{2} \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^2 + V(|\mathbf{u}|) - \mathbf{F} \cdot \mathbf{u} \right\}. \quad (2)$$

After introducing the new variables $\mathbf{x}' = \mathbf{x}/\sqrt{\epsilon}$ and $\tau' = \tau/\sqrt{\rho}$, the Euclidean action (2) can be rewritten in the form

$$S_{\text{Eucl}}[\mathbf{u}(\mathbf{x}', \tau')] = \rho^{1/2} \epsilon^{d/2} \int_{-\hbar/2\sqrt{\rho}T}^{+\hbar/2\sqrt{\rho}T} d\tau' \int_{\Omega'} d\mathbf{x}' \left\{ \frac{1}{2} \left(\frac{\partial \mathbf{u}}{\partial \tau'} \right)^2 + \frac{1}{2} \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}'} \right)^2 + V(|\mathbf{u}|) - \mathbf{F} \cdot \mathbf{u} \right\}. \quad (3)$$

Below we shall work with this Euclidean action and find its zero- and finite temperature instantons.

B. Zero temperature instanton

Let us calculate the Euclidean action corresponding to the extremal trajectories at a given temperature T . The variation of the action (3) yields the saddle-point equation

$$\frac{\partial^2 \mathbf{u}}{\partial \tau'^2} + \Delta' \mathbf{u} = \frac{\partial V}{\partial \mathbf{u}} - \mathbf{F}. \quad (4)$$

We need to find the solution satisfying the boundary condition $\mathbf{u}(\mathbf{x}', 0) = \mathbf{u}(\mathbf{x}', \hbar/\sqrt{\rho}T)$. The solution of Eq. (4) can be written in the form $\mathbf{u} = (u, \dots, 0)$. At zero temperature the instanton is spherically symmetric and introducing the new variable $r^2 = t'^2 + \mathbf{r}'^2$ we obtain an ordinary differential equation for the function $u(r)$,

$$\frac{1}{r^d} (r^d u')' = u'' + \frac{d}{r} u' = \frac{\partial V}{\partial u} - F. \quad (5)$$

The function $u(r)$ must be non-zero and continuous, have a continuous derivative $u'(r)$, and satisfy the boundary condition

$$u(r), u'(r) \rightarrow 0, \quad r \rightarrow \infty. \quad (6)$$

The corresponding solution consists of two parts: a “macroscopic” solution for $r \lesssim R_d$ (R_d is a parameter which will be determined later) and a “microscopic” solution for $r \gtrsim R_d$. The former corresponds to displacements $u \gtrsim a$ (we remind the reader that a is the characteristic radius of the pinning potential), while the latter describes the bounce in the region $u \lesssim a$, where the potential is relevant.

Consider the “microscopic” solution where we can neglect the (small) force F in the equation of motion (5) as compared to the potential term $\partial_u V$. Furthermore, in the limit of small F the radius R_d of the nucleus is large (we will check this assumption at the end), such that we can neglect the term du'/r in Eq. (5), and the remaining equation takes the form

$$u'' = \frac{\partial V}{\partial u}. \quad (7)$$

Eq. (7) is identical to Newton’s second law. Accounting for the boundary condition (6) we obtain

$$\frac{u'^2}{2} - V(u) = 0, \quad (8)$$

which is equivalent to the law of energy conservation for the 1D conservative motion in classical mechanics. As $V(u \gtrsim a) \simeq V_0$, we can obtain the boundary condition for the function $u(r)$

$$u'(R_d) = -\sqrt{2V_0}. \quad (9)$$

A simple integration then provides the microscopic part of the solution, once the parameter R_d is known.

Next, we solve Eq. (5) in the region $r < R_d$ with the boundary conditions given by Eq. (9) and $u(R_d) \sim a$. As we will see below, $u(0) \propto 1/F$, allowing us to use the condition $u(R_d) = 0$ without changing the main term in the asymptotics of the Euclidean action in the limit $F \rightarrow 0$. In the region $r < R_d$ we can neglect the term $\partial V/\partial u$ in Eq. (5) and the general solution can be written in the form

$$u(r) = -\frac{Fr^2}{2(d+1)} + C_1 + C_2 f(r), \quad (10)$$

where

$$f(r) = \begin{cases} \frac{1}{r^{d-1}}, & d \neq 1, \\ \ln r, & d = 1. \end{cases} \quad (11)$$

The term $C_2 f(r)$ is always singular at the point $r = 0$, hence $C_2 = 0$. Taking into account the boundary condition (9) we obtain

$$R_d = \frac{\sqrt{2V_0}(d+1)}{F} \quad (12)$$

and the quantum instanton can be written in the form

$$u(r) = \frac{F}{2(d+1)} (R_d^2 - r^2). \quad (13)$$

Substituting the function $u(r)$ into Eq. (3) and taking into account that $r^2 = \tau'^2 + \mathbf{x}'^2$ we obtain the zero temperature Euclidean action

$$S_{\text{Eucl}}(T=0, d) = \frac{2^{(d+3)/2} A_{d+1}}{(d+3)} (d+1)^d \left(\frac{\sqrt{\epsilon V_0}}{F} \right)^d \sqrt{\rho V_0} \frac{V_0}{F} + O\left(\frac{1}{F^d}\right), \quad (14)$$

where $A_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of a unit sphere in d -dimensional space.

Neglecting the terms F or $\partial V/\partial u$ in Eq. (5) might seem questionable as there is an interval with the length of order u_F where $\partial V/\partial u \Big|_{u_F} \simeq F$. However, if this interval is much smaller than R_d the approximation used above is valid: Assume that the potential $V(u)$ at large u behaves as $V(u) \simeq V_0 - B/u^\alpha$, $\alpha > 0$. The equation $V' = F$, $F \rightarrow 0$ can be easily solved and yields the result $u_F \simeq (\alpha B/F)^{1/(\alpha+1)}$, i.e., in the limit $F \rightarrow 0$, $u_F \ll R_d$ and the approximation made above is asymptotically correct.

C. Thermal instanton

Next, let us calculate the activation energy. The saddle-point solution is time-independent in this case, i.e., we can eliminate the term $\partial^2 \mathbf{u}/\partial \tau'^2$ in Eq. (4). The remaining equation is identical to the $(d-1)$ -dimensional zero temperature problem considered above and we can use the instanton (13) with $d \rightarrow d-1$. We substitute this solution into Eq. (3) and obtain the expression for the activation energy,

$$U(d) = \frac{2^{(d+2)/2} A_d}{(d+2)} d^{d-1} \left(\frac{\sqrt{\epsilon V_0}}{F} \right)^d V_0 + O\left(\frac{1}{F^{d-1}}\right). \quad (15)$$

We remind the reader that for the thermal instanton $S_{\text{Eucl}}(T \gg T_c, d) = \hbar U(d)/T$.

Next, let us calculate the two characteristic temperatures $T_c(d)$ and $T^*(d)$. The former corresponds to the value of T , where the quantum and thermal exponents $S_{\text{Eucl}}(d)/\hbar$ and $U(d)/T$ match up, while the latter defines the limit of applicability of the zero temperature instanton (see Fig. 2): at temperatures $T > T^*(d)$ the instanton (13) does not satisfy the periodic boundary condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}(\mathbf{x}, \hbar/T)$. Note that the thermal instanton is always a valid solution as it is time-independent and satisfies the periodic boundary condition identically.

The exponents $S_{\text{Eucl}}(d)/\hbar$ and $U(d)/T$ are equal at the temperature $T_c(d)$ given by the expression (we make use of Eqs. (14) and (15))

$$T_c(d) = \frac{1}{\sqrt{2\pi}} \frac{d+3}{d(d+2)} \left(\frac{d}{d+1} \right)^d \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \frac{\hbar F}{\sqrt{\rho V_0}}. \quad (16)$$

Formally, the quantum instanton found in Section IIB is valid only at zero temperature, however, we can use it also at $T \neq 0$ as long as the periodic boundary conditions are satisfied asymptotically. The zero temperature bounce solution then can be still applied if the periodicity in imaginary time τ' , $\hbar/\sqrt{\rho} T$, is larger than the diameter of the nucleus $2R_d$, i.e., if $R(d) < \hbar/2\sqrt{\rho} T$ (see Fig. 2).

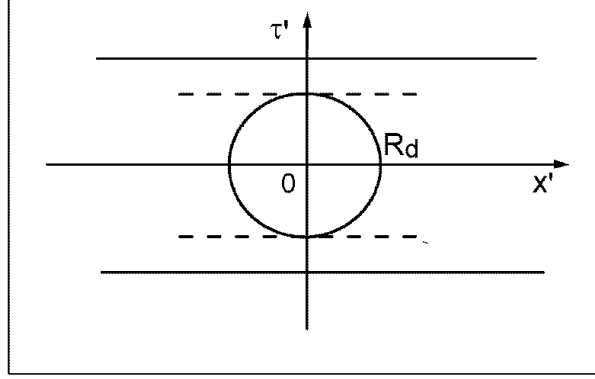


Fig.2 Nucleus corresponding to the low temperature decay process. As long as the diameter $2R_d$ remains smaller than the periodicity in time $\hbar/\sqrt{\rho}T$ (solid lines) the instanton (13) asymptotically satisfies the boundary conditions. At temperatures $T > T^*(d) = \hbar/2\sqrt{\rho}R_d$ the boundary conditions are not fulfilled and the distortion of the nucleus' walls becomes relevant.

Making use of Eq. (12) we obtain the following expression for the temperature $T^*(d)$ at which one needs to take into account the distortion of the quantum instanton due to the periodic boundary conditions,

$$T^*(d) = \frac{1}{2\sqrt{2}(d+1)} \frac{\hbar F}{\sqrt{\rho V_0}}. \quad (17)$$

III. INSTABILITY TEMPERATURE AND EUCLIDEAN ACTION

In this section we calculate the instability temperature $T_0(d)$ of the thermal instanton. At this temperature the saddle-point solution becomes time-dependent. Then we show that $T_0(d) < T_c(d) < T^*(d)$ for $1 \leq d \leq 6$ and, consequently, the problem exhibits a first-order transition from quantum to classical behavior.

Let us expand the Euclidean action around the thermal saddle-point solution. As the bounce satisfies the Euler-Lagrange equations, the term proportional to $\delta \mathbf{u}$ ($\delta \mathbf{u}$ describes the distortion of the saddle-point configuration) is equal to zero. To second order in $\delta \mathbf{u}$ we obtain

$$\begin{aligned} S_{\text{Eucl}}[\mathbf{u}_{th}(\mathbf{x}') + \delta \mathbf{u}] &\simeq S_{\text{Eucl}}[\mathbf{u}_{th}(\mathbf{x}')] + \frac{1}{2} \int_{-\hbar/2\sqrt{\rho}T}^{+\hbar/2\sqrt{\rho}T} d\tau' \int_{\Omega'} d\mathbf{x}' \delta u_1 \left(\hat{H}_1 - \rho \frac{\partial^2}{\partial \tau'^2} \right) \delta u_1 \\ &+ \frac{1}{2} \int_{-\hbar/2\sqrt{\rho}T}^{+\hbar/2\sqrt{\rho}T} d\tau' \int_{\Omega'} d\mathbf{x}' \delta \mathbf{u}_{\perp} \left(\hat{H}_{\perp} - \rho \frac{\partial^2}{\partial \tau'^2} \right) \delta \mathbf{u}_{\perp}, \end{aligned} \quad (18)$$

where $\delta u_1 = \delta \mathbf{u} \cdot (1, 0 \dots 0)$ and $\delta \mathbf{u}_{\perp} = \delta \mathbf{u} - \delta u_1 \cdot (1 \dots 0)$. The operator \hat{H}_1 can be written in the form

$$\hat{H}_1 = -\Delta' + \frac{\partial^2 V}{\partial u^2} \Big|_{u_{th}}. \quad (19)$$

The instability temperature is determined by its only negative eigenvalue. The operator \hat{H}_{\perp} has only positive eigenvalues.

Let us calculate the lowest eigenvalue $-\omega^2$ of the operator (19). Here, we follow the procedure described in detail in Refs. 13,15,17. First, we show that the operator (19) has d zero eigenvalues: the d functions $\phi_i = \partial u_{th} / \partial x'_i$, $i = 1, \dots, d$, satisfy the equation $\hat{H}_1 \phi_i = 0$. Due to the spherical symmetry of the instanton and, consequently, of

the “potential” in Eq. (19), the eigenfunctions of the d -dimensional Schrödinger operator (19) can be represented in the form $\phi_i = \Psi_i(\tilde{r})Y_i(\Omega) = \partial_{\tilde{r}}u_{th}\partial_{x'_i}\tilde{r}$, with $\Psi_i(\tilde{r})$ and $Y_i(\Omega)$ the radial and angular components, respectively, $\tilde{r}^2 = \sum_{i=1}^d x'_i{}^2$. For the d eigenfunctions corresponding to the zero eigenvalue $\Psi_i(\tilde{r}) = \Psi(\tilde{r}) = \partial_{\tilde{r}}u_{th} \propto \tilde{r}$ for $\tilde{r} < R_{d-1}$ (see Eq. (13) and the first paragraph in section II C). In the region $\tilde{r} \gtrsim R_{d-1}$ the eigenfunctions of the lowest levels are asymptotically ($F \rightarrow 0$) equal, allowing us to find the boundary condition for the lowest eigenvalue $-\omega^2$ from that of the zero eigenvalue wavefunction. The condition for binding the radial parts of the wavefunctions at the point $\tilde{r} = R_{d-1}$ takes the form

$$\left. \frac{\Psi'(\tilde{r})}{\Psi} \right|_{\tilde{r}=R_{d-1}} = \frac{1}{R_{d-1}}. \quad (20)$$

The general solution of the equation

$$\hat{H}_1\Psi_{-1} = -\omega^2\Psi_{-1} \quad (21)$$

in the region $\tilde{r} < R_d$ can be written in the form

$$\Psi_{-1}(\tilde{r}) = \tilde{r}^{(2-d)/2} [C_1 I_{(d-2)/2}(\sqrt{\omega}\tilde{r}) + C_2 K_{(d-2)/2}(\sqrt{\omega}\tilde{r})], \quad (22)$$

where $I_\nu(z)$ and $K_\nu(z)$ are Bessel functions of imaginary argument. The function $K_{(d-2)/2}(z)$ is singular at the point $z = 0$ and hence $C_2 = 0$. (For the case $d = 1$ this analysis is not valid, however it can be easily shown that the equation for the lowest eigenvalue (see Eq. (23)) remains the same). Using Eq. (20) we obtain a transcendental equation for the negative eigenvalue $-\omega^2(d) = -4\pi^2\rho T_0^2(d)/\hbar^2$,

$$\frac{\mu_d I'_{(d-2)/2}(\mu_d)}{I_{(d-2)/2}(\mu_d)} = \frac{d}{2}, \quad \mu_d = \omega(d)R_{d-1}, \quad (23)$$

and

$$T_0(d) = \frac{\mu_d}{2\sqrt{2}\pi d} \frac{\hbar F}{\sqrt{\rho V_0}}. \quad (24)$$

Using Eqs. (16) and (17) and the definition of the parameter μ_d , see Eq. (23), we find

$$\chi_d \equiv \frac{T_c}{T^*} = \frac{2}{\sqrt{\pi}} \frac{(d+1)(d+3)}{(d+2)d} \left(\frac{d}{d+1} \right)^d \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \quad (25)$$

and

$$\frac{T_0(d)}{T_c(d)} = \frac{d+1}{d} \frac{\mu_d}{\pi\chi_d}. \quad (26)$$

TABLE I. Ratios $T_c(d)/T^*(d)$ and $T_0(d)/T_c(d)$ for $1 \leq d \leq 6$. For $T_0(d) < T_c(d) < T^*(d)$ the system exhibits a sharp transition from quantum to classical behavior. For $d = 7$, both ratios exceed unity and no conclusion can be drawn regarding the order of the transition.

dimensionality, d	μ_d	$\chi_d = T_c(d)/T^*(d)$	$T_0(d)/T_c(d)$
1	1.1997	$\frac{8}{3\pi} \approx 0.8488$	0.8998
2	1.6083	$\frac{5}{6} \approx 0.8333$	0.9215
3	1.9150	$\frac{27}{10\pi} \approx 0.8594$	0.9457
4	2.1725	$\frac{112}{125} \approx 0.8960$	0.9647
5	2.3993	$\frac{5000}{1701\pi} \approx 0.9357$	0.9795
6	2.6048	$\frac{32805}{33614} \approx 0.9759$	0.9918

In Table I we summarize the numerical values of the parameters χ_d and $T_0(d)/T_c(d)$. We see that for $1 \leq d \leq 6$ the relation $T_0(d) < T_c(d) < T^*(d)$ holds and, consequently, the Euclidean action has the form plotted in Fig. 3: It is equal to $S_{\text{Eucl}}(d)$ (see Eq. (14)) for $T < T_c(d)$ (see Eq. (16)), whereas for $T > T_c(d)$ we have $S_{\text{Eucl}}(d, T) = \hbar U(d)/T$. The jump from the zero-temperature to the high-temperature instanton takes place at $T = T_c$: the system exhibits a sharp transition from quantum to classical behavior, see Fig. 3.

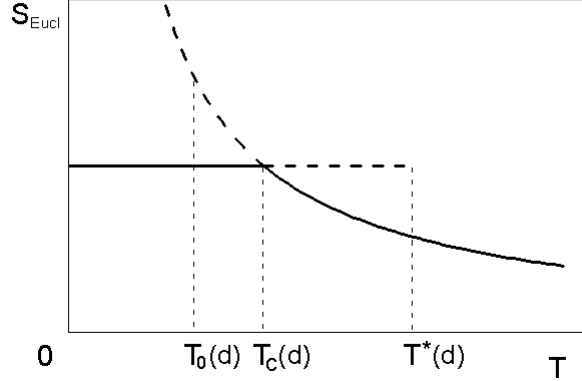


Fig.3 The Euclidean action as a function of temperature for $1 \leq d \leq 6$. For $T < T_c$ the Euclidean action is constant and given by Eq. (14) (see Eq. (16) for an expression for T_c). For $T > T_c$ the action is equal to $\hbar U(d)/T$, with $U(d)$ given by Eq. (15).

IV. PREEXPONENTIAL FACTOR FOR THE (1+1)-PROBLEM

In this section we calculate the preexponential factor for the (1 + 1)-problem in the high temperature region, improving on the results of Ref. 16. We start from Langer's expression for the decay rate¹⁸ (see also Ref. 19)

$$\hbar\Gamma = 2T_0(1)\frac{\text{Im}Z}{Z}, \quad (27)$$

where $T_0(1) = T_0(d=1)$, see Eq. (24) and Table I, and Z is the partition function of the system. For a (1+1)-string we can write^{15,20}

$$\hbar\Gamma = \frac{T_0(1)L}{\sqrt{2\pi T}} \left[\int_{-\infty}^{+\infty} dx \left(\frac{\partial u_{th}}{\partial x} \right)^2 \right]^{1/2} \left| \frac{\det(\delta^2 S / \delta u^2|_{u=0})}{\det'(\delta^2 S / \delta u^2|_{u=u_{th}(r)})} \right|^{1/2} \exp\left(-\frac{U(1)}{T}\right), \quad (28)$$

where L is the length of the string and the prime indicates that we exclude the zero eigenvalue of the operator $\hat{H}_1 = \delta^2 S / \delta u^2|_{u=u_{th}(x)}$. The activation energy $U(1)$ is given by the expression

$$U(1) = \frac{4\sqrt{2}}{3} \sqrt{\epsilon V_0} \frac{V_0}{F} \quad (29)$$

and the eigenvalues of the operators $\hat{H}_0 \equiv \delta^2 S / \delta u^2|_{u=0} = -\partial_x^2 + \partial_u^2 V|_{u=0}$ and $\hat{H}_1 \equiv \delta^2 S / \delta u^2|_{u=u_{th}(x)} = -\partial_x^2 + \partial_u^2 V|_{u_{th}}$ take the form

$$\lambda_{0,\alpha n} = \lambda_{0,\alpha} + \rho \left(\frac{2\pi T}{\hbar} n \right)^2, \quad n = 0, \pm 1, \pm 2 \dots \quad (30)$$

and

$$\lambda_{1,\alpha n} = \lambda_{1,\alpha} + \rho \left(\frac{2\pi T}{\hbar} n \right)^2, \quad n = 0, \pm 1, \pm 2 \dots, \quad (31)$$

respectively. Carrying out the product over n we obtain the high temperature result

$$\hbar\Gamma = \frac{T_0(1)L}{\sqrt{2\pi T}} \left[\int_{-\infty}^{+\infty} dx \left(\frac{\partial u_{th}}{\partial x} \right)^2 \right]^{1/2} \left| \frac{\det \hat{H}_0}{\det' \hat{H}_1} \right|^{1/2} \exp \left(-\frac{U^*(1)}{T} \right), \quad (32)$$

where

$$U^*(1) = U(1) - \frac{\hbar}{2\pi\sqrt{\rho}} \left[\int_0^{k^*} dk \delta\rho(k) \sqrt{\kappa + \epsilon k^2} - \sum_{\alpha} \sqrt{\lambda_{1,\alpha}} \right] \quad (33)$$

is the quantum renormalized activation energy, see Ref. 15. In Eq. (33) $\kappa = V''(0)$, $\delta\rho(k)$ is the difference in the continuous part of the spectral densities of the operators \hat{H}_0 and \hat{H}_1 , and the sum is taken over the positive discrete eigenvalues of the operator \hat{H}_1 (note that the operator \hat{H}_0 has no discrete eigenvalues). At large k^* the correction to $U(1)$ diverges as $\ln k^*$, i.e., we have to introduce a cutoff k^* into the problem. The determinant ratio in Eq. (32) has been calculated by Krämer and Kulić¹⁷ using the Gelfand-Yaglom formula^{21,22} with the result

$$\left| \frac{\det \hat{H}_0}{\det' \hat{H}_1} \right| \sim \frac{F}{a} \exp \left(\frac{\sqrt{2\kappa V_0}}{F} \right). \quad (34)$$

The coefficient of proportionality depends on the detailed form of the pinning potential $V(u)$. The final answer for the decay rate then takes the form

$$\Gamma \sim FL \left(\frac{1}{\rho a T} \sqrt{\frac{V_0}{\epsilon}} \right)^{1/2} \exp \left\{ \frac{\sqrt{2\kappa V_0}}{F} - \frac{U^*(1)}{T} \right\}; \quad (35)$$

accounting for gaussian fluctuations we obtain an exponential enhancement of the decay rate. Note that $U^* \sim 1/F$, see Eqs. (33) and (29). The result (35) is valid for temperatures T below the kink activation energy E_k

$$T \lesssim \frac{FU^*(1)}{\sqrt{2\kappa V_0}} \sim a\sqrt{\epsilon V_0} \sim E_k \quad (36)$$

(here we assumed that the quantum correction to the activation energy $U(1)$ is small and $\kappa \sim V_0/a^2$). At higher temperatures we need to take into account the renormalization of the free energy due to thermal fluctuations: The free energy per unit length of an elastic string trapped in a potential well coincides with the ground state energy of a massive particle in the equivalent quantum mechanical problem²³, where the temperature plays the role of Planck's constant. If the temperature is small, the ground state energy is close to zero. At high temperatures, satisfying the condition $V_0 \lesssim T^2/\epsilon a^2$ (which is the condition for a weak potential well, see Ref. 24), the ground state energy increases substantially and becomes comparable with the potential well depth. In this case the thermal instanton found in Sec. II is not applicable and the problem is more conveniently solved in the quantum mechanical formulation: Since in the 1D quantum particle trapped in an attractive potential always exhibits a bound state, the $(1+1)$ -dimensional string problem at $F = 0$ does not undergo a depinning transition at any temperature.

It is instructive to compare the result (35) to that obtained for the case of thermally activated motion of a string between two nearly degenerate minima^{15,25,26}, see Fig. 1(a), where the prefactor shows a power-like dependence on the external force F . The different dependences of the prefactors follow from the different low energy spectra: In the present case we have $\sim V_0/Fa$ low-lying eigenvalues of order F^2/V_0 (see Ref.²⁷), the product $P(F)$ of which behaves as $\ln P(F) \sim -1/F$, while for the problem studied in Refs. 15,25,26 only one small eigenvalue exists¹¹.

Finally, let us compare the result (35) with that obtained previously by Skvortsov¹⁶ where the semiclassical approximation for the eigenvalues of the operators \hat{H}_0 and \hat{H}_1 has been used. The derivation in Ref. 16 differs from the present one in two respects: *i*) The calculation has been based on the low temperature expression

$$\hbar\Gamma = 2T \frac{\text{Im} Z}{Z} \quad (37)$$

for the decay rate, such that the final high temperature result lacks the correct classical limit ($\Gamma \propto 1/\hbar$, see Eq. (57) in Ref. 16). *ii*) Correcting for the factor T_0/T , the remaining difference can be traced back to the use of the quasiclassical approximation in the calculation of the ratio (35), which produces a prefactor $\propto F^{3/2}$ rather than the correct result $\propto F$. Note that at zero temperature Eq. (37) is applicable, however, the semiclassical approach is not accurate enough to produce the correct expression for the decay rate (though it is still possible to obtain the correct exponential

enhancement of the preexponential factor and the correct quantum renormalized Euclidean action S^* , see Ref. 16, similar to the high temperature case discussed above). In principle, the complete prefactor in the low temperature quantum regime can be calculated using the procedure suggested in Refs. 28,29: The calculation of the determinant ratios can be reduced to the calculation of an infinite product of determinants of 1D Schrödinger operators, which then can be calculated numerically.

V. CONCLUSION

As possible applications of the present problem we discuss *i)* the thermal depinning of vortices in high- T_c superconductors and *ii)* the nucleation of phase transitions in the vicinity of a boundary or an interface.

i) The dynamics of vortices in high-temperature superconductors is dominated either by the dissipative or the Hall term. The vortex mass can be neglected. However at high temperatures the activation energy is independent of the dynamics (see Eq. (15)) and we can use Eq. (15) for the description of the thermal depinning of a single vortex line ($d = 1$) from a columnar defect ($n = 2$), or of a vortex sheet ($d = 2$) from an interface between superconducting grains³⁰ ($n = 1$). We wish to point out that Eq. (15) can also be used for the activation energy of a single vortex depinning from a twin boundary. Note that a vortex pinned by a columnar defect ($(1 + 2)$ -dimensional problem) again does not undergo a depinning transition at any finite temperature: the corresponding 2D quantum mechanical problem always has a bound state and hence the vortex remains localized at all temperatures provided that $F = 0$, see Ref. 23. As our result is restricted to temperatures below the kink energy, our analysis is not in contradiction with this result.

ii) Consider the nucleation of a new phase⁷ in the vicinity of a boundary¹⁶ or interface. Let us assume that there exist two competing (D -dimensional) phases, the stability of which are depending on the given external conditions. Initially, phase 1 is prepared, after which the external conditions are changed adiabatically, rendering phase 1 metastable. The nucleation of phase 2 takes a finite time and might be of two types: usual bulk nucleation or nucleation in the vicinity of the boundary. The latter can be described as the problem of depinning of a $((D - 1) + 1)$ -dimensional manifold. The boundary between the two phases plays the role of the elastic manifold, while the difference in the chemical potentials plays the role of the small external force. It is interesting to study the competition between these two possible types of nucleation. The free energy \tilde{U}_D of the D -dimensional spherical nucleus of radius R is given by the expression

$$\tilde{U}_D(R) = \epsilon A_D R^{D-1} - F V_D R^D, \quad (38)$$

where $A_D = 2\pi^{D/2}/\Gamma(D/2)$ and $V_D = 2\pi^{D/2}/D\Gamma(D/2)$ are the surface and the volume of a unit sphere in D -dimensional space. Minimizing this energy with respect to R we obtain

$$R = \frac{(D - 1)\epsilon}{F} \quad (39)$$

and

$$\tilde{U}_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \frac{(D - 1)^{D-1}}{D} \epsilon \left(\frac{\epsilon}{F}\right)^{D-1}. \quad (40)$$

This expression should be compared to $U_D \equiv U(D - 1)$ as given by Eq. (15). After some algebra we obtain

$$\frac{\tilde{U}_D}{U_D} = \frac{\sqrt{\pi}}{2^{(D+1)/2}} \frac{(D^2 - 1)}{D} \frac{\Gamma((D - 1)/2)}{\Gamma(D/2)} \left(\frac{\epsilon}{V_0}\right)^{(D+1)/2}. \quad (41)$$

It is reasonable to assume that $\epsilon \approx V_0$, i.e., the boundary tension ϵ roughly matches up with the pinning potential V_0 . In this case $\tilde{U}_D/U_D > 1$ for $D = 2$ and 3 , i.e., nucleation at the boundary is more favorable than that in the bulk (the same result applies also for the weak pinning case $V_0 \ll \epsilon$). The analysis for the quantum case is more complicated: The kinetic energy of the nucleus can be written in the form $f(R)\dot{R}^2$, with $f(R)$ a function of the nucleus' radius R . The process of nucleation of the new phase leads to a redistribution of mass in space as the phases have different densities in general. For the case $D = 3$ the function $f(R)$ has been calculated in Ref. 7, where it has been assumed that both phases are incompressible quantum liquids. For the case $D = 2$ this approach gives a divergent kinetic energy and, consequently, we need a more detailed description of the phases 1 and 2 to reach a sensible result. Similarly, the effective mass per area of the boundary between the two phases cannot be calculated assuming incompressible phases.

Briefly summarizing, we have analyzed the problem of depinning of a $(d + n)$ -dimensional massive elastic manifold from an $O(n)$ -invariant trapping potential $V(\mathbf{u}) = V(|\mathbf{u}|)$ in the presence of a small external force \mathbf{F} . For the case $1 \leq d \leq 6$ the Euclidean action has been calculated in the whole temperature range, see Eq. (14) for $T < T_c$ (T_c is given by Eq. (16)) and Eq. (15) for $T > T_c$, and we have found a sharp transition from quantum to classical behavior, see Fig. 3. The high-temperature asymptotics for the preexponential factor of the $(1 + 1)$ -problem has been calculated. Possible applications to the thermal depinning of vortices in high- T_c superconductors and the nucleation problem in the vicinity of a boundary in D -dimensional phase transitions have been discussed and we have shown that for $D = 2 - 4$ nucleation at the boundary is more favorable than from the bulk.

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